Discrete Curves

Warmup: Smooth Setting

Univariate curves
WARMUP: SMOOTH SETTING

Univariate curves
- secant
- tangent
- circle
- curvature
- signed curvature

\[ \kappa = \frac{1}{r} \]
**Gauß Map:** $\tilde{n}(s)$

Map to unit circle

- **Shape operator**
  
  $S = d\tilde{n}$
  
  $S(\tilde{t}) = \langle d\tilde{n}(\tilde{t}), \tilde{t} \rangle$
  
  $S\left(\frac{d}{ds}c(s)\right) = \kappa(s)$

**Turning Number**

Number of orbits in Gauß image

- Different homotopy classes

+1, -1, +2, 0
**Turning Number Thm.**

For a closed curve

\[ \int_C \kappa ds = k \, 2\pi \]

**Discrete Setting**
Inscribed Polygon: \( p \)

Finite number of vertices
- on curve, ordered
- straight edges

Length

Sum of edge lengths

\[
l(p) = \sum_{i=1}^{n} l_i
\]
**Length**

Smooth curve
- limit of inscribed polygon lengths

\[
\sup_{p} l(p)
\]

**Total Signed Curvature**

Sum of turning angles

\[
T_K = \sum_{i=0}^{n} \alpha_i
\]
Discrete Gauß Map

Edges map to points, vertices map to arcs

Turning number well-defined for discrete curves
**Turning Number Theorem**

Closed curve
- the total signed curvature is an integer multiple of $2\pi$.
- proof: sum of exterior angles

\[
T_\kappa = \sum_{i=1}^{n} \alpha_i = k2\pi
\]

**Structure-Preservation**

Arbitrary discrete curve
- total signed curvature obeys discrete turning number theorem
- even on a coarse mesh
- can be crucial
  - depending on the application
Convergence

Consider refinement sequence
- length of inscribed polygon to length of smooth curve
- discrete measure approaches continuous analogue
- which refinement sequence?
  - depends on discrete operator
  - pathological sequences may exist

Recall: Total Signed Curvature

Sum of turning angles

\[ T_K = \sum_{i=0}^{n} \alpha_i \]
Another Definition

Curvature normal

\[ \kappa \hat{n} \]

signed curvature (scalar)  unit normal (vector)

Curvature Normal

Gradient of length

- define discrete curvature

\[ \nabla L = \kappa \hat{n} \]
GRADIENT OF LENGTH

\[ \nabla L_1 \]

\[ \pi - \theta \]
Gradient of Length

\[ \nabla L_2 + \nabla L_1 \]

\[ \pi - \theta \]
\[ \nabla L = \kappa \hat{n} = 2 \sin \frac{\theta}{2} \hat{n} \]
Moral of the Story

Structure-preservation

For an arbitrary (even coarse) discrete curve, the discrete measure of curvature obeys the discrete turning number theorem.

Convergence

In the limit of a refinement sequence, discrete measures of length and curvature agree with continuous measures.